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Comparing various exponentiated distributions through suitable illustration

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Abstract
Discussions were held about extended exponentiated weibull, extended exponentiated exponential, extended exponentiated lognormal, and extended exponentiated gumble distributions. Contrary to the weibull model, the two parameter exponentiated weibull can fit unimodel, monotone, and risk functions. Exponentiated exponential may be utilized as an alternative to weibull distribution and, in certain situations, has a better fit than weibull due to its shape and scale parameters. Better fits may be obtained using the unimodel distributions 3 parameter exponentiated lognormal and 2 parameter exponentiated gumble. For lifetime data, the extended exponentiated exponential distribution gives more flexibility. Maximum likelihood estimation is utilized for parameter estimation. Statistics for goodness of fit are shown for a collection of data.

Keywords: exponential, weibull, lognormal, gumble, maximum likelihood estimator, exponentiated exponential, goodness of fit

Introduction
The weibull distribution is most prevalent distribution for analyzing life time data. The study of survival data, mostly from the fields of engineering, agriculture, veterinary medicine, and medicine, has often utilized the weibull set of distributions. With the suggested exponentiated weibull distribution, Mudhokar *et al.* (1995) [2] and Mudhokar and Srivastava (1993) [1] have produced a number of failure time data sets. When applied to the weibull and exponential families of distributions, A. Marshall and I. Olkin's innovative approach for including a parameter to a family of distributions was first presented in 1997. A particular example of the generic class of exponentiated distributions suggested by Gupta *et al.* is the exponentiated weibull distribution (1998). Exponentiated exponential distribution may be utilized as an alternative to the two parameter gamma and weibull distribution, according to Gupta and Kundu's (2001) [4] presentation of the distribution and analysis of different lifetime data sets. Exponential gumble distribution for the survival function was first developed by Nadarajah (2005) [5]. He gave an example of how it may be used to predict rainfall data from Orland, Florida. Exponentiated lognormal distribution suited real-world data sets better than weibull and exponentiated exponential distribution, according to analyses by Kakade and Shirke (2006) [6]. Exponentiated lognormal distribution is regarded as a good substitute. To evaluate data that were favourably skewed, Kakade and Shirke (2007) [7] explored the exponential gumble distribution. On the extension of various exponentiated distributions with applications, Raja and Mir (2011) [8] made a contribution. Effective estimate of the cdf and pdf of the exponentiated gumble distribution was the focus of Bagheri *et al.* 2016' [10] research (2014). The exponentiated exponential distribution was extended by Abu-Youssef *et al.* (2015) [9], providing a more practical model for lifetime data sets. The shortened life test for the exponentiated exponential distribution was developed by Suresh and Usha in 2016. In 2019, Raja and Maqbool expanded the use of Poisson and Poisson Type distributions. The exponentiated gumble distribution has a superior fit and may be utilized instead of the weibull distribution, according to Malik Mansoor and Kumar Devendra's (2020) [13] study of the weibull model. Exponentiated exponential distribution offers a comparably better fit in the current investigation than three parameter weibull and exponentiated gumble.
In this study, we take into account each of these distributions individually and compare those using actual data.

Exponentiated Weibull Distribution
Probability density function

The "exponentiated weibull distribution's probability density function (P.d.f.) as examined by Mudhokar *et al.* with the parameters $\alpha, \theta, and \sigma$ and life time having a function like,

$$f(t;\alpha,\theta,\sigma)=\frac{\alpha\theta}{\sigma}\left[1-\exp(-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\exp\left[-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\left(\frac{t}{\sigma}\right)^{\alpha-1},\nabla t>0........(1)$$

Where α >0, θ>0 are shape parameters and σ >0 is a scale parameter.

When θ=1 and when α=1 and θ=1, it follows the exponential distribution and the weibull distribution, respectively.

The random variable T's survival function with exponentiated-Weibull density is as follows:

$$S(t;\alpha,\theta,\sigma)=P(T\geq t)=1-\left[1-\exp\left(-(\frac{t_i}{\sigma})^{\alpha}\right)\right]^{\theta}..... \quad (1.2)$$

The model can easily fit survival data.

Maximum Likelihood Estimators

Let $x_1, x_2, x_3......x_n$ a sample drawn at random from EW, the log probability may be as L (α,θ,σ)=n. log (αθ/σ)+(θ-1).

$$\sum_{i=1}^{n}\log(g(Ti))-\sum_{i=1}^{n}(Ti/\sigma)^{\alpha}+(\alpha-1).\sum_{i=1}^{n}\log(Ti/\sigma)......(1.3)$$

Where,

g(Ti)=g(Ti;α,θ)=1-exp(-T/σ)$^{\alpha}$

We may distinguish (1.1) in terms of three parameters.

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}+(\theta-1).\sum_{i=1}^{n}g_{\alpha}(Ti)/g(Ti)-\sum_{i=1}^{n}(Ti/\sigma)^{\alpha}.\log(Ti/\sigma)+\sum_{i=1}^{n}\log(T/\sigma)=0...(1.4)$$

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log(g(Ti))=0............ \quad (1.5)$$

$$\frac{\partial L}{\partial\sigma}=-\left(\frac{n\alpha}{\sigma}\right)+(\theta-1).\sum_{i=1}^{n}g_{\sigma}(Ti)/g(Ti))+(\alpha/\sigma).\sum_{i=1}^{n}(Ti/\sigma)^{\alpha}........ \quad (1.6)$$

Where,

$$g_{\alpha}(Ti)=\exp(-(Ti/\sigma)^{\alpha}.(Ti/\sigma)^{\alpha}.\log(Ti/\sigma),$$

$$g_{\sigma}(Ti)=-\left[\alpha.\exp(-(Ti/\sigma)^{\alpha}).(Ti/\sigma)^{\alpha}\right]/\sigma$$

From (1.4), (1.5) and (1.6) we achieve the ML Estimates.

Exponentiated Exponential
Probability density function

Gupta and Kunda (2001) [4] defined density function with the parameters λ and α as

$$f(x,\alpha,\lambda)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}.e^{-\lambda x}...... \quad (2.1)$$

Where $\alpha,\lambda,x>0$

The shape parameter in this example is α and the scale parameter is λ. When α =1, the exponential family is represented.

The survival function is given as

$$S(x,\alpha,\lambda)=1-\left(1-e^{-\lambda x}\right)^{\alpha}....... \quad (2.2)$$

An parallel system is represented as an exponential with exponents.

Maximum Likelihood Estimators

Let $x_1, x_2, x_3......x_n$ be a random sample from EE the log likelihood can be as.

$$L(\alpha,\lambda)=n \text{ in } \alpha+n \text{ in } \lambda+(\alpha-1)\sum_{i=1}^{n}In(1-e^{-\lambda xi})-\lambda\sum_{i=1}^{n}x_i...... \quad (2.3)$$

Therefore in order to get MLE's of α and λ we can maximize (2.3) with respect to α and λ or we solve the non-linear normal equations as:-

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}In\left(1-e^{-\lambda x}\right)=0....... \quad (2.4)$$

$$\frac{\partial L}{\partial\lambda}=\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}}-\sum_{i=1}^{n}x_i=0....... \quad (2.5)$$

We obtain the MLE of α as a function of λ, say α ($\lambda^{\wedge}$), from (2.4) as

$$\alpha(\lambda^{\wedge}),=-\frac{n}{\sum_{i=1}^{n}In(1-e^{-\lambda x_i})}...... \quad (2.6)$$

The" MLE of the shape parameter may be $\alpha^{\wedge}$ derived simply from the scale parameter if it is known (2.6). If both parameters are unknown, it is possible to immediately maximise L(α ($\lambda^{\wedge}$), λ) with respect to λ get the scale parameter's estimate first. When $\lambda^{\wedge}$ is achieved $\alpha^{\wedge}$, it is also possible to do such as from (2.6) as α^ ($\lambda^{\wedge}$),

Exponentiated Lognormal Distribution $\lambda^{\wedge}$
Probability density function

With regard to three parameters (α,μ,σ), the exponential lognormal distribution's density function is defined "as

$$F(x;(\alpha,\mu,\sigma))=\alpha\left(\varphi(In(x);\mu,\sigma)\right)^{\alpha-1}.\phi\left(In(x);\mu,\sigma\right).x^{-1},...... \quad (3.1)$$

x, α >0, $-\infty<\mu<\infty$

Where, $\varphi(In(x);\mu,\sigma)$ and $\phi(In(x);$◊ are the C.d. f and P.d.f of the normal distribution with mean and standard deviation as μ and σ.

The following is the survival function that corresponds to the exponentiated lognormal distribution density:

S(x, μ σ. α) =1- ($\varphi(In(x);\mu,\sigma)$) $^{\alpha}$

Where x >0

Maximum Likelihood Estimators

Let $x_1, x_2, x_3......x_n$ a random sample drawn from the EL distribution the following is the log likelihood function:

$$L(\alpha,\mu,\sigma/x) = nIn\alpha - \sum_{i=1}^{n} In(xi) + (\alpha-1)\sum_{i=1}^{n} In\varphi(In(xi);\mu,\sigma) + \sum_{i=1}^{n} In\phi(In(xi);\mu,\sigma) \quad ...(3.2)$$

We solve the following equations to get the values of the parameters α,μ,σ that maximize L (α,μ,σ /x):

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} In\varphi(In(xi);\mu,\sigma) = 0....... \quad (3.3)$$

$$\frac{\partial L}{\partial \mu} = (\alpha-1)\sum_{i=1}^{n}\frac{\varphi_{\mu}^{/}(In(xi);\mu,\sigma)}{\varphi(In(xi);\mu,\sigma)} + \sum_{i=1}^{n}\frac{\varphi_{\mu}^{/}(In(xi);\mu,\sigma)}{\phi(In(xi);\mu,\sigma)} = 0.........(3.4)$$

$$\frac{\partial L}{\partial \sigma} = (\alpha-1)\sum_{i=1}^{n}\frac{\varphi_{\sigma}^{/}(In(xi);\mu,\sigma)}{\varphi(In(xi);\mu,\sigma)} + \sum_{i=1}^{n}\frac{\varphi_{\sigma}^{/}(In(xi);\mu,\sigma)}{\phi(In(xi);\mu,\sigma)} = 0.........(3.5)$$

From (3.3), (3.4) and (3.5) MLE of α, μ and σ is obtained respectively.

Exponential Gumble Distribution
Probability density function

The Probability density function (P.d.f) of exponential gumble distribution
Introduced by Nadarajah (2005) [5] with parameters α and σ is

$$f(x;\alpha,\sigma) = \frac{\alpha}{\sigma}\left[\exp\left(-e^{-\frac{x}{\sigma}}\right)\right]^{\alpha}.e^{-\frac{x}{\sigma}}, ... \quad (4.1)$$

Where α, σ>0 and $-\infty < x < \infty$
Where the scale parameter is σ and the shape parameter is α.
When α=1 in this case, the distribution eases to the standard gumble.

The provided survival function is

$$S(x,\alpha,\sigma) = 1 - \frac{\alpha}{\sigma}\left(\exp(-e^{\frac{x}{\sigma}})\right)^{\alpha} \quad .... \quad (4.2)$$

The α th power of the gumble distribution's survival function is the survival function of the exponentiated gumble distribution.

Maximum Likelihood Estimators

Let $x_1, x_2, x_3......x_n$ be a random sample from EG distribution the log likelihood function is

$$L(\alpha,\sigma) = n\ln\alpha - n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}x_i - \alpha\sum e^{\frac{-xi}{\sigma}} .... \quad (4.3)$$

Thus to achieve the MLE's of α and σ we can maximize (3.3) with respect to α and σ or and solve the non-linear normal equations as:-

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} = 0......... \quad (4.4)$$

$$\frac{\partial L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}x_i - \frac{\alpha}{\sigma^2}\sum_{i=1}^{n}x_i.e^{\frac{-x_i}{\sigma}} = 0..... \quad (4.5)$$

From (4.4) and (4.5) MLE of α, and σ is attained.

5. Extended Exponentiated Exponential Distribution:-

Let X be a random variable with survival function F¯(x), the survival function is as:

G¯(x) = λF¯(x) / (1 − λ¯F¯(x)), −∞ < x < ∞, λ > 0, λ¯ = 1−λ. (5.1)

Abu-Youssef *et al.* (2015) [9] introduced a new variant of the Marshall-Olkin extended family of distributions by selecting in (5.1) the exponentiated exponential distribution with survival function which yields

G¯(x) = λ − λ (1 − e −βx) α λ + λ¯ (1 − e −βx) α, x > 0, α > 0, λ > 0, β > 0 ……. (5.2)

Maximum likelihood estimators

Let X1, X2, Xn is a random sample of size n from the Marshall OLKIN Extended Exponentiated Exponential distribution then the likelihood function is:-

Ln (α, β, λ)=$\prod_{i=1}^{n}$ g(xi, α, β, λ) = ∏ $\frac{\alpha\beta\lambda(1 - e - \beta xi).\alpha - 1.e - \beta xi}{(\lambda^{-}(1 - e - \beta xi)\ \alpha + \lambda)\ 2}$…. (5.3)

and the log- likelihood function will be

Ln =$\sum_{i=1}^{n}$(α − 1)(log(1 − e − βxi)) − βxi − 2 log ((λ¯ (1 − e −βxi) α + λ)) + n log (αβλ) …… (5.4)

The Maximum Likelihood Estimation (MLE) of α, β and λ are obtained from
∂L/∂α = 0,
∂L/∂β = 0, and
∂L/∂λ = 0.

Goodness of Fit

Eight models gumble, weibull, exponentiated weibull, lognormal, exponentiated lognormal, exponentiated exponential, exponentiated gumble and extended exponentiated exponential were applied to real time data sets. Following are the distributions and the pdf:

Distribution P.d.f

Weibull f(x, α, λ) = αλ $(x\lambda)^{\alpha-1}.e^{(-\lambda x)^{\alpha}}$; α, λ, x >0

Lognormal f(x, μ, σ) = $\frac{\exp\left(-\frac{1}{2}\left(\frac{In(x)-\mu}{\sigma}\right)^2\right)}{x.\sigma.\sqrt{2\lambda}}$; $-\infty < \mu < \infty, \sigma > 0$

Gumble f(x, σ) $= \frac{1}{\sigma}\exp^{\frac{-x}{\sigma}}.\exp\left(-e^{\frac{-x}{\sigma}}\right)$; σ>0

Exponentiated weibull $f(t;\alpha,\theta,\sigma)=\frac{\alpha\theta}{\sigma}\left[1-\exp(-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\exp\left[-\left(\frac{t}{\sigma}\right)^{\alpha}\right]\left(\frac{t}{\sigma}\right)^{\alpha-1}$, t >o

Exponentiated exponential $f(x,\alpha,\lambda)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}.e^{-\lambda x},\quad \alpha,\lambda,x>0$

Exponentiated lognormal $f(x,(\alpha,\mu,\sigma))=\alpha\left(\varphi(In(x);\mu,\sigma)\right)^{\alpha-1}.\phi\left(In(x);\mu,\sigma\right).x^{-1}$, x, α >0, $--\infty<\mu<\infty$

Exponentiated gumble

$$f(x;\alpha,\sigma)=\frac{\alpha}{\sigma}\left[\exp\left(-e^{-\frac{x}{\sigma}}\right)\right]^{\alpha}.e^{-\frac{x}{\sigma}},\ \alpha,\sigma>0$$

Extended Exponentiated exponential f(x) = αβλ(1 − e −βx) α−1 e −βx (λ ̄(1 − e −βx) α + λ) 2, x > 0.

Data Set 1

The data" pertains to survival times (in days) of 36 bacilli infected lambs.

0.59, 0.51, 0.63, 0.77, 0.91, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07,.08,.08, 1.08,1.09, 1.09, 1.12, 1.13, 1.61, 1.72, 1.66, 1.83, 1.95, 1.93, 1.90, 2.02, 2.13, 2.23, 2.16,2.18, 2.30, 2.31, 2.40, 2.45, 2.53

Table 1: Distribution along with MLE's, Log-likelihood and Anderson's value.

Distribution	MLE'S	Log likelihood	Anderson's Value
Weibull	$\hat{\alpha}$=0.674, $\hat{\lambda}$=0.018	-142.75	0.069
Lognormal	$\hat{\mu}$=2.352, $\hat{\lambda}$=1.292	-143.37	0.065
Gumble	$\hat{\alpha}$=22.94, $\hat{\lambda}$=45.84	-141.34	0.062
Exponentiated weibull	$\hat{\alpha}$= 2.456,$\hat{\theta}$= 0.237 $\hat{\sigma}$=23.87	-139.11	0.057
Exponentiated exponential	$\hat{\alpha}$=0.762, $\hat{\lambda}$=0.017	-147.63	0.079
Exponentiated lognormal	$\hat{\alpha}$= 0.115 $\hat{\mu}$= 3.56 $\hat{\sigma}$=0.453	-143.82	0.055
Exponentiated gumble	$\hat{\alpha}$=1.57, $\hat{\lambda}$=42.56	-142.72	0.054
Extended exponentiated exponential	$\hat{\alpha}$=0.765, β= 138, $\hat{\lambda}$=0.0145	-129.65	0.051

The following tables gives a comparison between the MLE's Log-likelihood, and" Anderson's statistics.

Conclusion

We compared the exponentiated weibull, exponentiated lognormal, extended exponentiated exponential, exponentiated exponential, and exponentiated gumble probability density functions and their applications to a set of data. Contrary to the weibull model, the 2 parameter exponentiated weibull can fit unimodel, monotone, and risk functions. Exponentiated exponential may be used as a replacement for weibull distribution and, in many situations, has a better fit than weibull due to its similar shape and scale parameters. The unimodel distributions 2 parameter exponentiated gumble and 3 parameter exponentiated lognormal may provide superior fits. The extended exponentiated exponential distribution with three parameters provides a more adaptable model for real-time data sets. Exponentiated exponential provides a superior match for the data set, followed by weibull. Therefore, in certain circumstances, they may be substituted as alternatives to one another. One can try other suitable distributions depending upon the flexibility.

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